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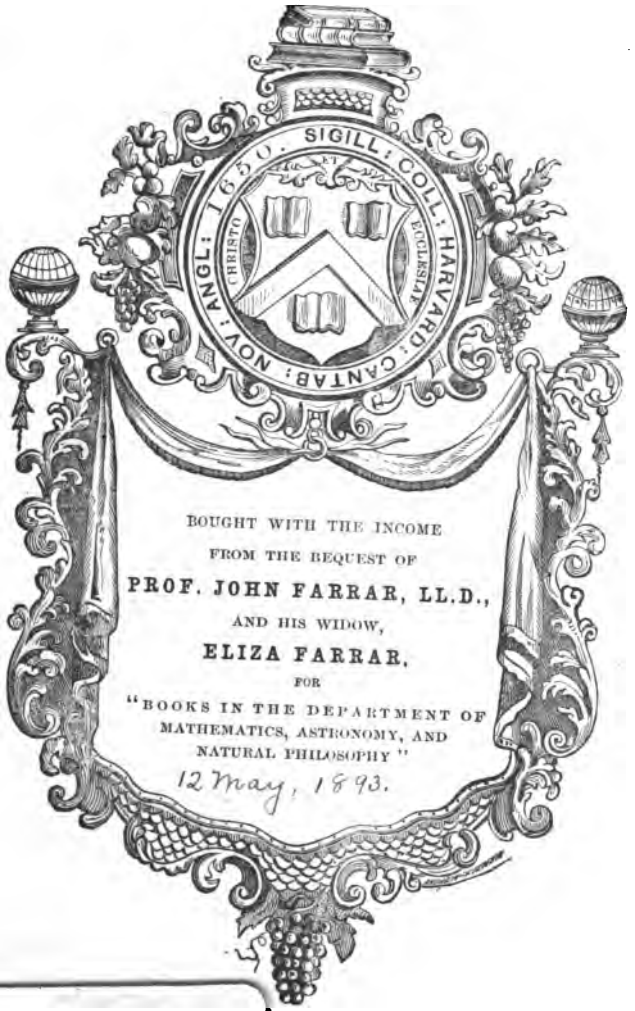

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MENSURATION.

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PREFACE.

This work is designed for the course in lower mathematics of schools and colleges.

It presupposes an elementary knowledge of Geometry and Trigonometry, and for the sake of brevity, repetition of definitions and demonstrations given in these branches have been omitted; where a principle is used that has been established in either of these branches, the student is referred to one of them for the proof.

As a very limited time is devoted to Mensuration in most Mathematical courses, the recent separate books on this subject are too extensive for convenient class-room use, while the older Trigonometries that contain Mensuration have been largely supplanted by more modern books that do not give any Mensuration. It is to meet the needs of institutions that have adopted these later Trigonometries, that this short course in Mensuration has been prepared. The object has been to give but little more than is commonly taught in the undergraduate courses of good colleges, affording a sufficient number of examples for practice.

EASTON, PA., January, 1893.

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MENSURATION.

§ 1. Mensuration

is that branch of mathematics which treats of the measurement of geometrical magnitudes. It is a branch of applied geometry and gives the rules for finding the lengths of lines, the areas of surfaces, and the volumes of solids. By Mensuration the values of these magnitudes are calculated from the simplest measurements.

§ 2. Units of Measure.

The *measure* of a geometrical magnitude is the ratio which it bears to a standard magnitude of the same kind taken as the unit of measure.

The *linear unit* may be an inch, a foot, a yard, or any given arbitrary length. If the unit's length is one foot, and the given line contains this unit's length, ten times, then the given line is ten feet long.

A *surface unit* is the square described upon a unit of length. The measure of a surface is called its area. If the surface unit is a square foot, then the area of any surface is the number of square feet which it contains.

A *volume unit* is a cube whose edge is a unit of length. If the unit for measurement of volumes is a cube whose edge is one foot, then the measure of any volume is the number of cubic feet which it contains.

The measures of geometrical magnitudes, being ratios, may commonly be expressed by abstract numbers. Hence, in the calculations of Mensuration, the numerical values of these magnitudes are taken.

In Mensuration, the expression *product of two lines*, denotes the product obtained by multiplying the number of linear units in one line by the number of units of the same kind in the other line. Thus, the area of a triangle is equal to one-half the product of the number of linear units in the base, by the number of linear units in the altitude.

The expression, *product of three lines*, denotes the product obtained by multiplying together the number of linear units in each of the three lines. Thus, the measure of the volume of a rectangular parallelopiped is equal to the number of linear units in one side of the base, multiplied by the number of linear units in the adjacent side, and this product multiplied by the number of linear units in the altitude.

§ 3. Commensurable and Incommensurable Magnitudes.

Two magnitudes are commensurable when they have a common measure; that is, when each will contain another magnitude an exact number of times. Thus, two straight lines whose lengths are five and seven inches are commensurable; for, each one contains one inch, the unit of measure, an exact number of times.

Two magnitudes are incommensurable when no common measure can be found. Thus, 1 and $\sqrt{2}$ are incommensurable; for $\sqrt{2}=1.4142+$, which is an interminable decimal, and no unit can be found which is contained in each an exact number of times. If we take $\frac{1}{10}$ as the unit of

measure, it will be contained 10 times in 1, an exact number of times; but it will be contained 14 times in $\sqrt{2}$, with a remainder of .0142 +. If, however, we take $\frac{1}{100}$ as the unit of measure, it will be contained 100 times in 1; but it will be contained 141 times in $\sqrt{2}$ with a remainder of .0042 +. Hence, by continually decreasing the unit of measure, we can get a fraction so small as to be contained in $\sqrt{2}$ a number of times, with a remainder as small as we please, or finally, less than any assignable quantity. Therefore, in what follows, only commensurable magnitudes will be considered.

§ 4. TABLES.

Table I. — Linear Measures.

Inches.	Gunter's Links.	Feet.	Yards.	Fathoms.	Rods.	Gunter's Chains.	Furlongs.	Mile.
$7\frac{92}{25}$ =	1							
12 =	$1\frac{17}{33}$ =	1						
36 =	$4\frac{6}{11}$ =	3 =	1					
72 =	$9\frac{1}{11}$ =	6 =	2 =	1				
198 =	25 =	$16\frac{1}{2}$ =	$5\frac{1}{2}$ =	$2\frac{3}{4}$ =	1			
792 =	100 =	66 =	22 =	11 =	4 =	1		
7,920 =	1,000 =	660 =	220 =	110 =	40 =	10 =	1	
63,360 =	8,000 =	5,280 =	1,760 =	880 =	320 =	80 =	8 =	1

Table IV. — Other Measures.

128	cubic feet (wood)	= 1 cord.
$24\frac{3}{4}$	cubic feet (stone)	= 1 perch.
282	cubic inches	= 1 ale gallon.
231	cubic inches	= 1 wine gallon.
2150.42	cubic inches	= 1 bushel.

Table II. — Square Measures.

Sq. Inches.	Sq. Links.	Sq. Feet.	Sq. Yards.	Sq. Rods.	Sq. Chains.	Roods.	Acres.	Sq. Mile.
$624\frac{4}{625}$ =	1							
144 =	$2\frac{232}{1089}$ =	1						
1,296 =	$20\frac{80}{121}$ =	9 =	1					
39,204 =	625 =	$272\frac{1}{4}$ =	$30\frac{1}{4}$ =	1				
627,264 =	10,000 =	4,356 =	484 =	16 =	1			
1,568,160 =	25,000 =	10,890 =	1,210 =	40 =	$2\frac{1}{2}$ =	1		
6,272,640 =	100,000 =	43,560 =	4,840 =	160 =	10 =	4 =	1	
4,014,489,600 =	64,000,000 =	27,878,400 =	3,097,600 =	102,400 =	6,400 =	2,560 =	640 =	1

Table III. — Cubic Measures.

Cu. Inches.	Cu. Feet.	Cu. Yards.	Cu. Rods.	Cu. Fur.	Cu. Mile.
1,728 =	1				
46,656 =	27 =	1			
7,762,392 =	$4,482\frac{1}{3}$ =	$166\frac{2}{3}$ =	1		
496,793,088,000 =	287,496,000 =	10,648,000 =	64,000 =	1	
2,544,358,061,056,000 =	147,197,952,000 =	5,451,776,000 =	32,768,000 =	512 =	1

MENSURATION OF LINES.

A. Lines in Plane Triangles.

§ 5. In every triangle there are six parts to be considered, namely, the three sides and the three angles. By Trigonometry, if any three of these parts are given, provided one part is a side, all of the other parts may be calculated and the triangle completely determined, and its area can be obtained.

§ 6. In a right-angled triangle, given the base and altitude, to find the hypothenuse.

[1] Rule: *Add the squares of the perpendicular sides and extract the square root of the sum.*

For Proof: See Ww. Geom., 331.

§ 7. In a right-angled triangle, given the hypothenuse and one side, to find the other side.

[2] Rule: *From the square of the hypothenuse subtract the square of the given side, and extract the square root of the difference.*

§ 8. In an oblique-angled triangle, when two sides and the included angle are given, to find the third side.

[3] Rule: *The side opposite the given angle is equal to the square root of the sum of the squares of the other two sides, diminished by twice the rectangle of these sides multiplied by the cosine of the included angle.*

In each of the triangles (Fig. 1 and Fig. 2), denote the side AB by c, BC by a, AC by b, and the angle BAC by A: and let AH be the projection of AB on CA produced.

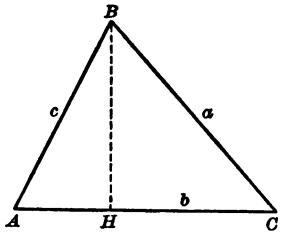

FIG. 1

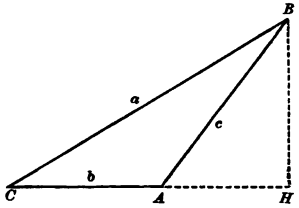

FIG. 2.

First Case. When the included angle is acute.

$$a^2 = c^2 + b^2 - 2b.AH. \qquad \text{(Fig. 1.)}$$

(Ww. Geom., 335.)

But in Fig. 1, $AH = c \cos A$.

$$\therefore\ a^2 = c^2 + b^2 - 2bc \cos A.$$

Second Case. When the included angle is obtuse.

$$a^2 = c^2 + b^2 + 2bAH. \qquad \text{(Fig. 2.)}$$

(Ww. Geom., 336.)

But in Fig. 2, $AH = c \cos BAH = -c \cos A$.

$$\therefore\ a^2 = c^2 + b^2 - 2bc \cos A.$$

Hence, in each case,

$$a = \sqrt{c^2 + b^2 - 2bc \cos A}. \qquad \text{Q.E.D.}$$

EXAMPLES.

1. The base of a right-angled triangle is 27 yards, and the altitude is 36 yards; what is the hypothenuse?

Ans., 45 yds.

2. The hypothenuse of a right-angled triangle is 315 feet and the base 299 feet; required the perpendicular.

Ans., 99.116 ft.

3. A ladder 30 feet long, placed near the middle of a street, reached the building on one side 24 feet from the ground; and with its foot at the same place, reached the building on the opposite side 18 feet from the ground. What was the width of the street? *Ans.*, 42 ft.

4. A vertical pole 90 feet high, standing on a horizontal plane was broken by the wind, the broken part resting on the upright, and the top on the ground 30 feet from the base; what was the length of the broken part?

Ans., 50 feet.

5. In an oblique-angled triangle, the given sides are 540 and 450, and the included angle is 80°; required the other side. *Ans.*, 640.082

6. Given the two sides, 1686 and 960, and the included angle 128° 04′; required the third side of the triangle.

Ans., 2400 yards.

7. The two given sides of an oblique-angled triangle are 3754 and 3277.628, and the included angle is 57° 53′ 17″; what is the third side? *Ans.*, 3428.512.

B. Angular Measurement and Arcs of Circles.

§ 9. In measuring angles a constant angle is taken as the unit. A convenient unit is a right angle. The right angle is divided into 90 equal parts called degrees, and each degree into 60 equal parts called minutes, and each minute into 60 equal parts called seconds. Angles at the centre of a circle have the same ratio as the arcs which are subtended by them (Ww. Geom., 201). Thus, an angle at the centre may be said to be measured by its intercepted arc; that is, an angle at the centre is such part of four right angles or such part of the angular magnitude about

a point in a plane, as its intercepted arc is of the whole circumference.

Therefore, $\frac{\text{any angle}}{\text{4 right angles}} = \frac{\text{intercepted arc}}{\text{circumference}}$.

§ 10. Another unit angle that is often used is called a "*radian.*" A *radian* is the angle at the centre of a circle subtended by an arc equal to the radius. The radian measure of an angle is sometimes called its circular measure.

Let AN (Fig. 3) = arc $NC = r$;

then angle CAN = radian.

Let angle $NAM = 90°$;

then arc $NM = \frac{1}{4} \times 2\pi r$;

(Ww. Geom., 403.)

by § 9

$$\frac{\text{number of degrees in radian } CAN}{\text{number of degrees in } NAM} = \frac{\text{arc } NC}{\text{arc } NM} = \frac{r}{\frac{1}{2}\pi r} = \frac{2}{\pi} ;$$

$\therefore$ number of degrees in radian $= 90° \times \frac{2}{\pi} = 57.295779°+$.

§ 11. To find the number of radians in any angle.

RULE: *The number of radians in any angle is equal to the length of the arc intercepted, divided by the radius.*

Let BAN, Fig. 3, represent any angle at the centre, angle CAN being the radian, and let the length of the arc BN be S; then

$$\frac{\text{number of radians in } BAN}{\text{one radian}} = \frac{\text{arc } BN}{\text{arc } CN} = \frac{S}{r} ;$$

therefore, number of radians in $BAN = \frac{S}{r}$.

If $BAN = 360°$, then $S = 2\pi r$, and the number of radians in four right angles will be $\frac{2\pi r}{r} = 2\pi$; then in two right angles there will be π radians, and in one right angle, $\frac{1}{2}\pi$ radians.

§ 12. To pass from degrees to radians. Let x denote the number of degrees in an angle, and z the number of radians in the same angle; then, since there are π radians in two right angles,

$$\frac{x}{180^\circ} = \frac{z}{\pi}$$

or, $$z = \frac{\pi x}{180^\circ} = \text{number radians in angle } x.$$

§ 13. Given the radius of a circle, to find the circumference.

[4] RULE: *The circumference of a circle is equal to the radius multiplied by* 2π.

For Proof: See Ww. Geom., 403.

§ 14. Given the circumference of a circle to find the radius.

[5] RULE: *The radius of a circle is equal to the circumference divided by* 2π.

π is equal to $3.1415926535 +$, an interminable decimal. The approximate value commonly used is 3.1416. The value $3\frac{1}{7}$ is often taken where great accuracy is not required, but the fraction $\frac{355}{113}$ is much nearer the correct value.

§ 15. To find the length of an arc containing any number of degrees.

[6] RULE I: *The length of an arc is to the circumference of the circle as the number of degrees in the arc is to 360°.*

For Proof: See Ww. Geom., 200 and 201.

[7] RULE II: *The length of an arc is equal to the number of degrees in the arc, multiplied by the diameter of the circle, and this product by 0.0087266.*

Demonstration: The circumference of a circle whose diameter is unity is 3.1416. The length of an arc of *one* degree in this circle will be equal to the circumference, or 3.1416, divided by 360; that is, it will be equal to 0.0087266; hence, the length of an arc of n degrees, will be, $n \times 0.0087266$. And when the diameter is d, the length of the arc will be equal to the length when the diameter is unity multiplied by d (Ww. Geom., 375).

EXAMPLES.

1. What is the diameter of a circle whose circumference is 100 yards? *Ans.*, 31.831.

2. What is the circumference of a circle whose diameter is 25? *Ans.*, 78.54.

3. If the diameter of the earth is 7921 miles, what is its circumference? *Ans.*, 24,884.6136 miles.

4. If the circumference of a wheel is 16 feet 6 inches, what is its diameter? *Ans.*, 5.2522 ft.

5. If the circumference of the moon's orbit is 1,492,987 miles, what is its diameter? *Ans.*, 475,233 miles.

6. If the diameter of a circle is 17 chains, what is the circumference? *Ans.*, 53.4072 chains.

7. What is the length of an arc of 30°, in a circle whose diameter is 18 feet? *Ans.*, 4.712364 ft.

8. What is the length of an arc of 12° 10′, the diameter being 20 feet? *Ans.*, 2.123472 ft.

9. What is the length of one degree of longitude at the Equator, if the circumference of the earth is 24,856.28 miles? *Ans.*, 69.045 miles.

§ 16. To find the length of an arc, when the chord of the arc and the radius of the circle are given.

[8] RULE: *The sine of one-half the angle at the center which the arc subtends, is equal to one-half the chord of the arc divided by the radius.*

Proof: In Fig. 3, let BC be the given chord, and AM the radius perpendicular to it. In the right-angled triangle ABH,

$$BH = AB \sin BAH$$
(Ww. Trig., § 10.)
$$= AB \sin \tfrac{1}{2} BAC \qquad (1)$$
$$\therefore \sin \tfrac{1}{2} BAC = \frac{BH}{AB}. \qquad (2)$$

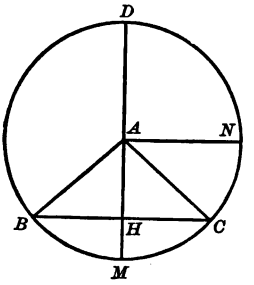

FIG. 3.

From equation (2) the angle BAC can be obtained; and the length of the arc may then be determined by § 15.

§ 17. To find the length of an arc, when the height of the arc and the radius of the circle are given.

[9] RULE: *The cosine of one-half the angle at the center which the arc subtends, is equal to the radius diminished by the height, and this difference divided by the radius.*

Proof: In the right-angled triangle ABH (Fig. 3),

$$AH = AB. \cos BAH \text{ (Ww. Trig., § 10).} \quad (1)$$

Now $$AH = AM - HM.$$

and $$BAH = \tfrac{1}{2} BAC$$

$$\therefore \cos \tfrac{1}{2} BAC = \frac{AM - HM}{AB}. \qquad (2)$$

From equation (2) the angle BAC can be obtained; and the length of the arc may then be determined by § 15.

§ 18. To find the height of an arc, when the chord of the arc and the radius of the circle are given.

[10] RULE: *From the radius subtract the square root of the difference of the squares of the radius and half the chord.*

Proof: In Fig. 3, $HM = AM - AH$,

and, $$AH^2 = AB^2 - BH^2,$$

$$\therefore\ HM = AM - \sqrt{AM^2 - BH^2}.$$

EXAMPLES.

1. What is the length of an arc, whose chord is 30, in a circle whose radius is 20?

Solution: In Fig. 3, let BC represent the chord. Then by Formula, § 16,

$$\sin \tfrac{1}{2} BAC = \frac{BH}{AB}.$$

By applying logarithms,

$$\log \sin \tfrac{1}{2} BAC = \log BH - \log AB + 10.$$

$$\log BH = \log 15 = 1.176091$$

$$\log AB = \log 20 = \underline{1.301030}$$

$$\log \sin \tfrac{1}{2} BAC = \overline{9.875061}.$$

$$\therefore\ \tfrac{1}{2} BAC = 48^\circ\ 35'\ 25''$$

and $$BAC = 97^\circ\ 10'\ 50''.$$

2. In a circle whose diameter is 100 feet, what is the length of an arc whose chord is 40 feet?

3. The height of an arc is 10 feet, and the radius of the circle is 25 feet; required the angle of the arc, and also its length.

4. A chain, 100 feet in length, is stretched between two points on a curved rail, and the middle ordinate is then found on measurement to be three feet; if the curve is a circular arc, what will be the radius of curvature?

MENSURATION OF PLANE SURFACES.

C. Plane Polygons.

§ 19. To find the area of a triangle, when base and altitude are given.

[11] Rule: *The area of a triangle, is equal to one-half the product of its base and altitude.*

For Proof: See Ww. Geom., 324.

§ 20. To find the area of a triangle, when two sides and their included angle are given.

[12] Rule: *The area of a triangle is equal to one-half the product of two sides into the sine of their included angle.*

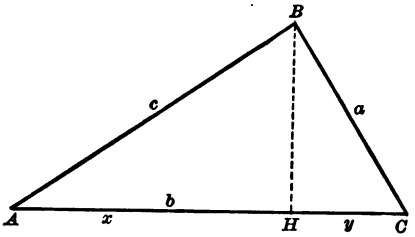

Fig. 4.

Proof: Let ABC, Fig. 4, represent a plane triangle, in which the side $AC = b$, $AB = c$, and A is the given included angle. From B draw BH perpendicular to AC; this will be the altitude of the triangle.

$$BH = c. \sin A. \text{ (Ww. Trig., § 10)}$$

$$\text{area } ABC = \tfrac{1}{2}\, AC \times BH. \text{ (§ 19)}$$

$$= \tfrac{1}{2}\, b.c. \sin A\ ;$$

or, $$2 \text{ area } ABC = b.c. \sin A\,.$$

Denoting the area by S, and applying logarithms, the equation becomes,

$$\log 2\ S = \log b + \log c + \log \sin A - 10.$$

§ 21. To find the area of a triangle when the three sides are given.

[13] RULE: *Find half the sum of the three sides, and subtract from it each side separately. Then find the continued product of the half sum and the three remainders, and extract its square root; the result will be the area of the given triangle.*

Demonstration: Let ABC, Fig 4, be a triangle whose sides a, b, and c are given, and let BH be a perpendicular from the vertex of the angle ABC to the opposite side, and represent the segments into which this perpendicular divides the base by x and y.

Since AHB and BHC are right-angled triangles,

$$c^2 = x^2 + BH^2 \text{ and } a^2 = y^2 + BH^2, \qquad (1)$$

solving equations (1) for BH^2, and equating, gives,

$$c^2 - x^2 = a^2 - y^2,$$

whence, $c^2 - a^2 = x^2 - y^2 = (x - y)\ (x + y) = b\ (x - y)$;

$$\therefore\ x - y = \frac{c^2 - a^2}{b}. \qquad (2)$$

but $$x + y = b \qquad (3)$$

solving equations (2) and (3) for x gives,

$$x = \frac{b^2 + c^2 - a^2}{2b}. \qquad (4)$$

Again,

$$BH^2 = c^2 - x^2 = c^2 - \left(\frac{b^2+c^2-a^2}{2b}\right)^2 = \frac{4c^2b^2-(b^2+c^2-a^2)^2}{4b^2}$$

$$= \frac{[2cb+(b^2+c^2-a^2)]\,[2cb-(b^2+c^2-a^2)]}{4b^2}$$

$$= \frac{[(c+b)^2-a^2]\,[a^2-(b-c)^2]}{4b^2}$$

$$= \frac{(b+c-a)\,(c+b+a)\,(a+c-b)\,(a+b-c)}{4b^2}. \quad (5)$$

Let $\frac{a+b+c}{2} = \frac{S}{2}$; then $\frac{c+b-a}{2} = \frac{1}{2}S - a$,

$$\frac{a+b-c}{2} = \frac{1}{2}S - c, \text{ and } \frac{a+c-b}{2} = \frac{1}{2}S - b;$$

substituting these values in (5), extracting the square root of each member, and reducing gives,

$$BH = \frac{2}{b}\sqrt{\tfrac{1}{2}S\,(\tfrac{1}{2}S-a)\,(\tfrac{1}{2}S-b)\,(\tfrac{1}{2}S-c)}. \quad (6)$$

Now by § 19, area $ABC = \frac{1}{2}\,b.\ BH.$

$$\therefore \text{ area } ABC = \sqrt{\tfrac{1}{2}S\,(\tfrac{1}{2}S-a)\,(\tfrac{1}{2}S-b)\,(\tfrac{1}{2}S-c)}. \quad (7)$$

Applying logarithms to equation (7), gives,

$$\log \text{ area } ABC = \tfrac{1}{2}\,[\log \tfrac{1}{2}S + \log(\tfrac{1}{2}S-a) + \log(\tfrac{1}{2}S-b) + \log(\tfrac{1}{2}S-c)].$$

§ 22. To find the area of an equilateral triangle.

If each of the sides is a, the expression for the area in Equation (7), § 21, becomes,

$$\text{Area } ABC = \sqrt{\tfrac{3}{2}a \times \tfrac{1}{2}a \times \tfrac{1}{2}a \times \tfrac{1}{2}a} = \frac{a^2\sqrt{3}}{4}.$$

EXAMPLES.

1. What is the area of a triangle whose base is 207.5 feet and height is 59.5 feet? *Ans.*, 6173.125 ft.

2. What is the area of a triangular field in acres, roods and perches, whose base is 24½ chains, and whose height is 18 chains? *Ans.*, 22 A., 0 R. 8 P.

3. What is the area of a triangle, two of whose sides are 30 and 40, and their included angle is 28° 57′?

Ans., 290.427.

4. What is the area of a triangle of which two sides are 45 and 32 feet, and the included angle 46° 30′?

Ans., 522.269.

5. What is the area of a triangle, whose sides are 20, 30, and 40?

Solution:

$$\tfrac{1}{2}S = \tfrac{1}{2}(20 + 30 + 40) = 45$$
$$\tfrac{1}{2}S - a = 45 - 20 = 25$$
$$\tfrac{1}{2}S - b = 45 - 30 = 15$$
$$\tfrac{1}{2}S - c = 45 - 40 = 5$$

$$\therefore \text{ area} = \sqrt{45 \times 25 \times 15 \times 5} = 290.4737 \text{ } \textit{Ans.}$$

6. What is the area of a triangle, whose three sides are, 23.7, 29.25, and 40.1 yards? *Ans.*, 343.3161 yds.

7. What is the area of an isosceles triangle whose base is 30 feet, and each of its equal sides 45 feet?

Ans., 636.3961 ft.

8. What is the area of a triangle whose sides are 125, 173, and 216 feet? *Ans.*, 10809 sq. ft.

9. What is the area of a triangle whose sides are each 37 feet? *Ans.*, 592.79 ft.

10. What is the area of an equilateral triangle whose sides are 22 perches? *Ans.*, 1 A., 1 R., 9.5781 P.

§ 23. To find the area of a parallelogram when the base and altitude are given.

[14] RULE: *The area of a parallelogram is equal to the product of its base and altitude.*

For Proof: See Ww. Geom., 321.

§. 24. To find the area of a trapezoid.

[15] RULE: *The area is equal to one-half the product of the altitude and the sum of the parallel sides.*

For Proof: See Ww. Geom., 327.

§ 25. To find the area of a parallelogram when two adjacent sides and the included angle are given.

[16] RULE: *The area of a parallelogram is equal to the product of the two adjacent sides, multiplied by the sine of the included angle.*

Proof: The area of a parallelogram is double the area of a triangle having the same two adjacent sides and included angle; but the area of the triangle is equal to one-half the product of the adjacent sides multiplied by the sine of the included angle (§ 20). Therefore the area of a parallelogram is as given by Rule [16].

§ 26. To find the area of a parallelogram when the diagonals and their included angle are given.

[17] RULE: *The area of a parallelogram is equal to one-half the product of the diagonals, multiplied by the sine of their included angle.*

Proof: Let $ABCD$, Fig. 5, be a parallelogram, and AC and BD the diagonals.

Then area $ABCD = 2$ area $ABC = 2\,(ABO + BOC)$
$= 2\,[\frac{1}{2}\,BO\,.\,AO.\ \sin\,AOB + \frac{1}{2}\,BO\,.\,CO.\ \sin\,COB)$. (§ 20)

But $$\sin AOB = \sin COB = \sin \phi .$$

$$\therefore \text{ area } ABCD = \tfrac{1}{2}[2\ BO\ (AO + CO) \sin \phi]$$

$$= \tfrac{1}{2}\ BD\,.\,AC\,.\sin \phi\ ;$$

or, $\log$ (2 area $ABCD$) $= \log BD + \log AC + \log \sin \phi - 10.$

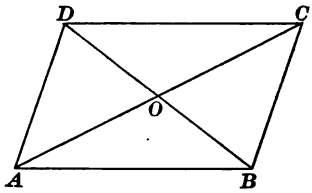

FIG. 5.

EXAMPLES.

1. How many square yards are there in a parallelogram whose base is 271 feet, and altitude 107 feet?

Ans., 3221⅜ sq. yds.

2. The area of a parallelogram is 4 acres, and the altitude 5 chains; required the length of the base.

Ans., 8 chains.

3. The area of a rectangle is 1472 yards, and the breadth is to the length as 3 to 4; required the sides.

Ans., 33.2264, and 44.3019 yds.

4. Required the area of a trapezoid whose two parallel sides are 25 feet 6 inches and 18 feet 9 inches, and the perpendicular distance between them 10 feet 5 inches.

Ans., 230$\frac{15}{32}$ sq. ft.

5. Required the area of a trapezoid whose two parallel sides are 750 and 1225 links, and the perpendicular distance between them 1540 links.

Ans., 15 A. 0 R. 32.2 P.

6. What is the area of a rhombus, each of whose sides is 21 feet 3 inches, and each of the acute angles 53° 20′?

Ans., 362.209 sq. ft.

7. What is the area of a parallelogram one of whose angles is 57° 25′, and the including sides are 125.81 and 57.65? *Ans.*, 6111.4.

8. The two diagonals of a parallelogram intersect each other at an angle of 30°, and their lengths are 25 and 72; required the area of the parallelogram.

9. Each diagonal of a rectangle is 20 feet, and the two diagonals intersect at an angle of 25°; what is the area of the rectangle?

§ 27. To find the area of any quadrilateral when one diagonal and the perpendiculars to it from the opposite vertices are given.

[18] RULE: *The area of any quadrilateral is equal to one-half the product of a diagonal and the sum of the perpendiculars to it from the vertices of the opposite angles.*

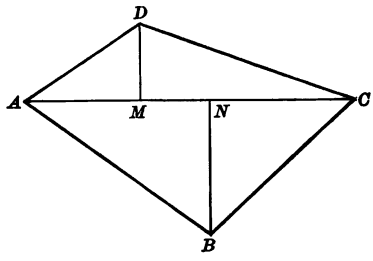

FIG. 6.

Proof: Let the diagonal AC in Fig. 6 divide the quadrilateral $ABCD$, into the triangles ABC, and ADC, of which the altitudes are respectively BN and DM.

Then area $ABCD$ = area ABC + area ADC

$= \frac{1}{2} AC \times BN + \frac{1}{2} AC \times DM$ (§ 19)

$= \frac{1}{2} AC\,(BN + DM)$.

§ 28. To find the area of a quadrilateral when the diagonals and their included angle are given.

[19] RULE: *The area of a quadrilateral is equal to one-half the product of its diagonals and the sine of their included angle.*

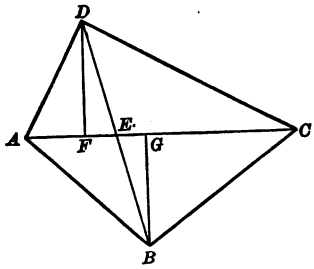

FIG. 7.

Proof: Let AC and DB, Fig. 7, be the diagonals of the quadrilateral $ABCD$, and let DF and BG be perpendiculars to AC, and let angle $BEC = a$. Then

area $ABCD$ = area ABC + area ADC.

Now $DF = DE \sin a$, and $BG = BE \sin a$.
(Ww. Trig., § 14.)

$\therefore$ area $ABCD = \frac{1}{2} AC.BE \sin a + \frac{1}{2} AC.DE \sin a$

$= \frac{1}{2} AC \times BD \sin a$.

§ 29. To find the area of any irregular polygon.

[20] RULE: *Divide the polygon into triangles, or into triangles and trapezoids by diagonals and perpendiculars to diagonals from vertices of the polygon; then find the sum of the areas of these separate figures.*

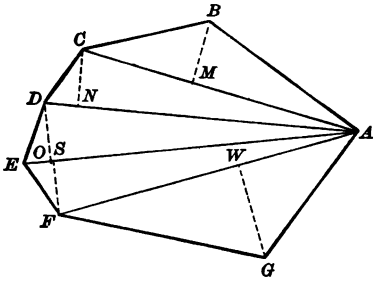

FIG. 8.

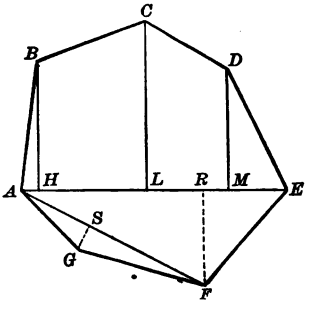

FIG. 9.

In Fig. 8 the area $ABCDEFGA$ = area ABC + area ADC + area ADE + area AEF + area AFG.

In Fig. 9 the area $ABCDEFGA$ = area ABH + area $BHLC$ + area $CLMD$ + area DME + area AFE + area AGF.

EXAMPLES.

1. Required the area of a trapezium, $ABCD$, Fig. 6, whose diagonal AC is 84 yards, the perpendicular BN, 28 yards, and the perpendicular DM 21 yards.

Ans., 2058 sq. yds.

2. How many acres are there in a quadrilateral whose diagonal is 80.5 chains, and the perpendiculars 22.4 and 28.3 chains? *Ans.*, 204 A. 0 R. 10.8 P.

3. How many square yards of paving are there in a quadrilateral surface whose diagonal is 65 feet, and the two perpendiculars let fall on it, from the opposite angles, 28 and 33½ feet? *Ans.*, 222.083 sq. yds.

4. In Fig. 8, $AC = 10$, $AD = 12$, $AE = 14$, $AF = 11$, $BM = 4$, $CN = 3$, $DS = 2$, $FS = 3$, and $GW = 5$; what is the area of the entire figure?

5. If the diagonals of a quadrilateral are 34 and 56, and if they intersect at an angle of 67°, what is the area?

Ans., 876.32.

6. If the diagonals of a quadrilateral are 75 and 100, and if they intersect at an angle of 34° 45′, what is the area?

7. In Fig. 9, $AH = 1$, $HB = 6$, $LC = 7$, $MD = 5$, $HL = 4$, $LM = 3$, $ME = 2$, $FR = 3$, $GS = 1$, and $AF = 8$; what is the area of the entire figure?

AREA OF POLYGONS BY THE METHOD OF COÖRDINATES.

§ 30. To find the area of a triangle in terms of the coördinates of its angular points.

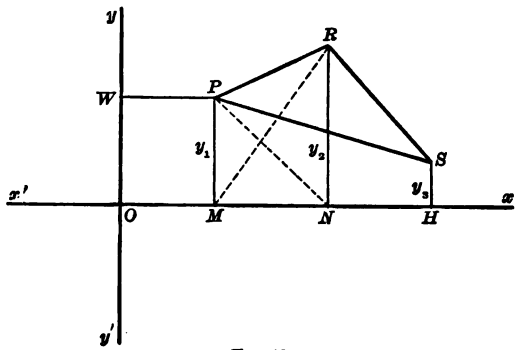

FIG. 10.

Two intersecting straight lines are regarded as the lines of reference and are called axes. The two coördinates of any point in the plane of these axes are its distances from the axes taken parallel to them; thus, in Fig. 10, the axes

are taken as XX' and YY', and the coördinates of the point P are $WP = OM$, and MP.

Let PRS be the given triangle, and the coördinates of the vertex P, OM and MP, be represented respectively by x_1 and y_1. Similarly represent the coördinates of R by x_2 and y_2, and of S by x_3 and y_3.

Now, $$MN = ON - OM = x_2 - x_1,$$
$$NH = OH - ON = x_3 - x_2,$$
and $$MH = OH - OM = x_3 - x_1,$$
also $$PRS = PMNR + RNHS - PMHS.$$
But $$PMNR = PMN + PNR = PMN + RMN,$$
and if the axes are perpendicular to each other,
$$PMNR = \tfrac{1}{2}\, PM.\, MN + \tfrac{1}{2}\, MN.RN = \tfrac{1}{2}\, MN\,(PM + RN)$$
$$= \tfrac{1}{2}\,(x_2 - x_1)\,(y_1 + y_2).$$
Similarly $$RNHS = \tfrac{1}{2}\,(x_3 - x_2)\,(y_2 + y_3),$$
and $$PMHS = \tfrac{1}{2}\,(x_3 - x_1)\,(y_1 + y_3);$$
hence $$\text{area } PRS = \tfrac{1}{2}\,[(x_2 - x_1)\,(y_1 + y_2)$$
$$+ (x_3 - x_2)\,(y_2 + y_3) - (x_3 - x_1)\,(y_1 + y_3)],$$
performing the indicated multiplications, canceling and factoring, gives

$$\text{area } PRS = \tfrac{1}{2}\,[y_1\,(x_2 - x_3) + y_2\,(x_3 - x_1) + y_3\,(x_1 - x_2)]. \quad (1)$$

Calling the coördinates parallel to the axis XX' abscissas, and those parallel to the axis YY' ordinates, the rule may be stated as follows:

[21] RULE: *The area of a triangle is equal to one-half the sum of the products of the ordinate of each vertex by the difference between the two adjacent abscissas.*

§ 31. To find the area of any polygon in terms of the coördinates of its angular points.

In Fig. 11 let $ABCDE$ represent the polygon, and let the coördinates of the vertices A, B, C, D and E be respectively (x_1, y_1) (x_2, y_2) (x_3, y_3) (x_4, y_4) and (x_5, y_5).

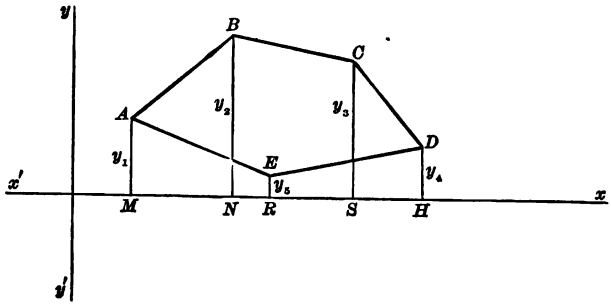

FIG. 11.

Now $ABCDE = ABMN + BCSN + CDHS - AMRE - DHRE = \frac{1}{2}(y_1 + y_2)(x_2 - x_1) + \frac{1}{2}(y_2 + y_3)(x_3 - x_2) + \frac{1}{2}(y_3 + y_4)(x_4 - x_3) - \frac{1}{2}(y_5 + y_1)(x_5 - x_1) - \frac{1}{2}(y_5 + y_4)(x_4 - x_5)$,

performing the indicated multiplications, canceling and factoring,

$$ABCDE = \tfrac{1}{2}[y_1(x_2 - x_5) + y_2(x_3 - x_1) + y_3(x_4 - x_2) + y_4(x_5 - x_3) + y_5(x_1 - x_4)]$$

and in general if n be the number of sides in any polygon, and if A represent the area,

$$A = \tfrac{1}{2}[y_1(x_2 - x_n) + y_2(x_3 - x_1) + y_3(x_4 - x_2) + \cdots y_n(x_1 - x_{n-1})]. \quad (1)$$

[22] RULE: *The area of any polygon is equal to one-half the sum of the products of the ordinate of each vertex by the difference between the two adjacent abscissas; always taking the differences in the same direction around the polygon.*

EXAMPLE.

In Fig. 12, let $ABCDE$ be a polygon, and OX and OY the axes. It is required to find the area from the following coördinates;

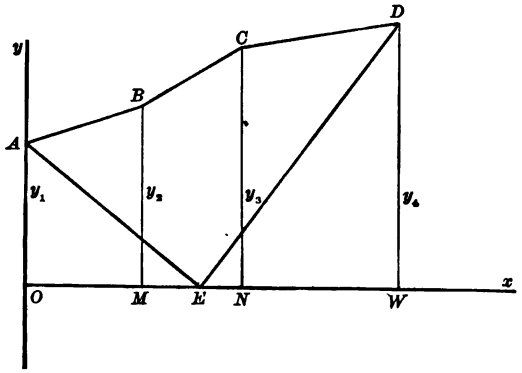

FIG. 12.

$$OA = y_1 = 10', MB = y_2 = 15',$$

$$NC = y_3 = 15', WD = y_4 = 16',$$

$$OM = x_2 = 9',\ ON = x_3 = 12',\ OW = x_4 = 18',$$

and $$OE = x_5 = 10'$$

substituting these values in formula (1) gives,

$$\text{area } ABCDE = \tfrac{1}{2}[10\,(9-10) + 12\,(12-0) + 15\,(18-9) + 16\,(10-12) + 0\,(0-18)]$$

$$= 118.5 \text{ square feet.} \quad \textit{Ans.}$$

§ 32. To find the area of a regular polygon.

[23] RULE: *The area of a regular polygon is equal to one-half the product of the perimeter and apothem.*

For Proof: See Ww. Geom., 379.

When the apothem is not given, it may be computed from the number of sides and length of one side. In Fig. 13, let $ABFHK$ be a regular polygon, whose centre is at C,

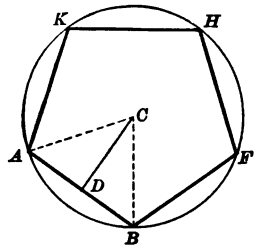

FIG. 13.

and whose apothem is CD. Denote the length of one side by s, and the number of sides by n.

The perpendicular CD bisects the side AB, and the angle ACB; and the angle ACB is equal to $\frac{360°}{n}$, (Ww. Geom., 370); therefore ACD will equal $\frac{180°}{n}$.

Then, in the right-angled triangle ACD,

$$CD = AD.\ \tan\ CAD \quad \text{(Ww. Trig., § 12)}$$

but, $\tan CAD = \cot ACD$; (Ww. Trig., § 5)

$$\therefore\ CD = \tfrac{1}{2}\, s.\ \cot \frac{180°}{n}. \qquad (1)$$

EXAMPLE.

In the regular pentagon, Fig. 13, let one side AB equal 25 feet; what is the area of the polygon?

Applying logarithms to equation (1) we have,

$$\log CD = \log (\tfrac{1}{2}\, s) + \log \cot \frac{180°}{n} - 10. \qquad (2)$$

$$\log\left(\tfrac{1}{2}s\right) = \log 12.5 \quad = \quad 1.096910$$

$$\log \cot \frac{180°}{n} = \log \cot 36° = \underline{10.138769}$$

$$\log CD = \underline{1.235649}$$

$$\therefore CD = 17.2048+$$

$$\text{area pentagon} = \frac{(5 \times 25) \times 17.2048+}{2} = 1075.4+ \quad \textit{Ans.}$$

By the preceding method the following areas of regular polygons have been computed on the supposition that each side is equal to a unit.

TABLE.

NAMES.	SIDES.	AREAS.
Triangle	3	0.4330127
Square	4	1.0000000
Pentagon	5	1.7204774
Hexagon	6	2.5980762
Heptagon	7	3.6339124
Octagon	8	4.8284271
Nonagon	9	6.1818242
Decagon	10	7.6942088
Undecagon.	11	9.3656399
Dodecagon	12	11.1961524

The areas of similar polygons are to each other as the squares of their homologous sides (Ww. Geom., 343).

Denoting the area of a regular polygon whose side is s by A_2, and the area of a similar polygon whose side is 1 by A_1

$$A_2 : A_1 :: s^2 : 1^2.$$

$$\therefore A_2 = A_1 s^2, \text{ hence,}$$

RULE II: *The area of a regular polygon is equal to the square of one of its sides multiplied by the area of a similar polygon whose side is unity.*

EXAMPLES.

1. Required the area of a regular hexagon whose side is 14.6 yards. *Ans.*, 553.632.

2. How many acres are contained in a regular heptagon whose sides are each 19.38 chains? *Ans.*, 135.66 acres.

3. The area of a regular decagon is 3233.4912 square feet; what is the length of the side? *Ans.*, $20\frac{1}{2}$ ft.

4. What is the area of a regular octagon, one of whose sides is 20? *Ans.*, 1931.36886.

5. What is the area of a regular dodecagon whose side is 54 feet? *Ans.* 32647.98 sq. ft.

D. Irregular Figure.

§ 33. To find the area of a plane figure with an irregular curvilinear perimeter (Fig. 14).

Take the line AN, as nearly as is convenient, in the general direction of the curved boundary. Then from the extremities, P_1, and M, let fall perpendiculars to AN. It is now required to find the area, $P_1 ANM$.

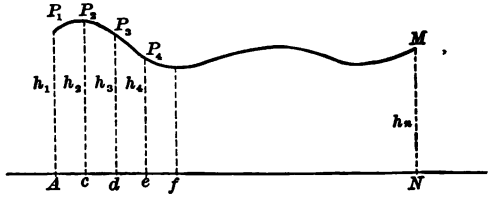

Fig. 14.

Divide AN into equal parts, and through the points of division c, d, e, etc., draw perpendiculars AP_1, cP_2, dP_3, etc.

Let $x = Ac = cd = de$ &c, and let $AP_1 = h_1$, $cP_2 = h_2$, &c.

$$P_1ANM = P_1AcP_2 + P_2cdP_3 + P_3deP_4 + \text{\&c}.$$
$$= \tfrac{1}{2}x(h_1 + h_2) + \tfrac{1}{2}x(h_2 + h_3) + \tfrac{1}{2}x(h_3 + h_4) + \text{\&c}.$$
$$= x[\tfrac{1}{2}(h_1 + h_n) + h_2 + h_3 + \cdots h_n].$$ Hence,

[24] RULE: *Find half the sum of the extreme perpendiculars, to which add the intermediate perpendiculars, and multiply this sum by the common distance between the perpendiculars.*

Evidently, this method gives only an approximation; but the greater the number of divisions, or in other words, the smaller the value given to x, the more accurately will the above rule give the required area.

EXAMPLE.

In Fig. 14, the base line AN is 100 feet, and is divided into ten equal parts, and beginning at A the following ordinates have been measured to the curve, 12, 11, 10, 9, 8, 8½, 9, 9, 8, 7, and 8; what is the area of the figure?

E. PLANE FIGURES CONTAINED BY CIRCULAR LINES.

§ 34. To find the area of a circle, when the radius is given.

[25] RULE I: *The area of a circle is equal to π, multiplied by the square of the radius.*

For Proof: See Ww. Geom., 381.

RULE II: *The area of a circle is equal to one-half the product of its radius by its circumference.*

For Proof: See Ww. Geom., 380.

§ 35. To find the radius of a circle when its area is given.

[26] RULE: *Divide the area by π and extract the square root of the quotient.*

EXAMPLES.

1. How many square feet are there in a circle whose diameter is 5 feet 6 inches? *Ans.*, 23.75835 sq. feet.

2. How many square yards are there in a circle whose radius is $15\frac{1}{4}$ feet? *Ans.*, 81.1798 sq. yards.

3. The area of a circle is $5028\frac{4}{7}$ square yards; what is its diameter? *Ans.*, 80.016 yards.

4. It is required to find the radius of a circle whose area is an acre. *Ans.*, 39.2507 yards.

5. The area of a circle is 2 acres, 3 roods, and 12 perches, what is the circumference? *Ans.*, 75.3637 sq. rods.

§ 36. To find the area of a sector of a circle.

[27] RULE I: *The area of a sector of a circle is equal to one-half the product of the arc of the sector and the radius.*

For Proof: See Ww. Geom., 382.

RULE II: *The area of the sector is equal to the area of the circle multiplied by the ratio of the arc of the sector to the circumference.*

EXAMPLE.

What is the area of a sector whose arc is 25° in a circle whose diameter is 10 feet?

$$\text{area circle} = \pi r^2 = 3.1416 \times 5^2 = 78.54$$

$$\therefore \text{area sector} = \tfrac{25}{360} \times 78.54 = 5.454 \quad \textit{Ans.}$$

§ 37. To find the area of a segment of a circle.

[28] RULE: *If the segment is less than a semicircle, subtract from the area of the sector which has the same arc, the area of the triangle formed by the chord of the segment and*

the radii of the sector; if the segment is greater than a semicircle, its area is equal to the sum of the areas of the triangle and sector.

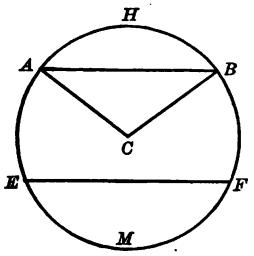

FIG. 15.

In Fig. 15 the segment AHB is equal to the sector $ACBH$, minus the triangle ACB; and the segment AMB is equal to the sector $ACBM$ plus the triangle ACB.

EXAMPLE.

What is the area of a segment whose arc contains 60° in a circle whose diameter is 10 feet?

$$\begin{aligned} \text{area circle} &= 3.1416 \times 5^2 &&= 78.54 \\ \text{area sector} &= \tfrac{60}{360} \times 78.54 &&= 13.09 \\ \text{area triangle} &= 25 \times 0.4330127 &&= 10.8253 + \\ &\therefore \text{area segment} &&= 2.2647 \quad \textit{Ans.} \end{aligned}$$

§ 38. To find the area of a circular zone.

A circular zone is that part of a circle which is included between two parallel chords.

[29] RULE: *The area of a circular zone is equal to the difference between the areas of two segments.*

As in Fig. 15 the zone $ABFE$ is equal to the segment EFH less the segment ABH.

EXAMPLES.

1. Required the area of a sector, the arc of which is 30° and the diameter 3 yards. *Ans.*, .58905 sq. yds.

2. Required the area of a sector, whose chord is $18\frac{3}{5}$ chains, and the diameter of the circle 20 chains.

Ans., 118.954 sq. chains.

3. The chord of the arc of a sector is 8 yards, and the height of the arc is 3 yards; what is the area of the sector?

Ans., 22.2222 sq. yards.

4. What is the area of the segment of a circle whose arc is 60°, and the diameter of the circle 10 feet?

Ans., 2.2647 sq. feet.

5. What is the area of a segment of a circle, whose height is 5 yards, and the diameter of the circle is 20 yards? *Ans.*, 61.1645 sq. yards.

6. What is the area of a circular zone, one of whose parallel chords is 30 feet, and the other 48 feet, and the distance between them 13 feet? *Ans.*, 534.19 sq. feet.

7. What is the area of a circular zone, the greater chord of which, being equal to the diameter of the circle, is 40 feet, and the less 20 feet? *Ans.*, 592.086 sq. feet.

§ 39. To find the area of a crescent.

A crescent is the surface included between the corresponding arcs of intersecting circles, and is that portion of one circle which is wholly external to the other.

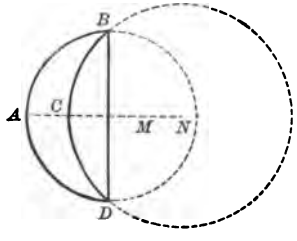

FIG. 16.

[30] RULE: *The area of a crescent is equal to the difference between the two segments on the same side of the common chord.*

In Fig. 16 area $ABCD$ = area ABD — area BCD.

EXAMPLES.

1. What is the area of a crescent whose chord is 20 feet; the height of one segment being 2 feet, and the diameter of the other circle being 52 feet?

2. Required the area of a crescent whose chord is 16, the diameter of one circle being $16\frac{2}{3}$, and the diameter of the other being 20?

Solution: area greater segment = 70.2222
area smaller segment = 44.764
$\therefore$ area crescent = 25.4582 *Ans.*

§ 40. To find the area of the surface included between two concentric circumferences.

[31] RULE: *The area of a ring included between two concentric circumferences is equal to the difference between the squares of the radii multiplied by* π.

Proof: Let the radius of the outer circle be R, and let the radius of the inner circle be r.

area greater circle $= \pi R^2$, (§ 34)
area smaller circle $= \pi r^2$,
$\therefore$ area ring $= \pi (R^2 - r^2)$.

EXAMPLES.

1. What is the area of a circular ring, the diameters of the concentric circles being 20 and 30 feet?

Ans., 392.7 sq. ft.

2. The diameters of two concentric circles are 8 and 12 yards; what is the area of the ring included between their circumferences? *Ans.*, 62.832 yds.

3. The radius of a circle is 3 chains, and the radius of an outer concentric circle is 10 chains; what is the area of the included ring? *Ans.*, 50.2656 sq. chains.

4. The diameters of two concentric circles are 320 and 280; what is the area of the ring included between their circumferences? *Ans.*, 18849.55.

ELLIPSE AND PARABOLA.

§ 41. To find the area of an ellipse.

[32] RULE: *The area of an ellipse is equal to the product of the semi-axes by* π.

EXAMPLE.

The axes of an ellipse are 66 and 22 yards; what is the area?

$$\text{Area} = \tfrac{66}{2} \times \tfrac{22}{2} \times 3.1416 = 1140.4 \text{ sq. yds.} \quad \textit{Ans.}$$

§ 42. To find the area of a parabola.

[33] RULE: *The area of a parabola is equal to two-thirds of the product of its base and height.*

EXAMPLE.

What is the area of a parabola whose base is 24 and height is 9?

$$\text{Area} = \tfrac{2}{3}\,(9 \times 24) = 144 \quad \textit{Ans.}$$

F. AREAS OF BROKEN AND CURVED SURFACES.

§ 43. To find the convex or lateral surface of a right cylinder or right prism.

[34] RULE: *The convex surface of a right cylinder or right prism is equal to the perimeter of the base multiplied by the altitude.*

For Proof: See Ww. Geom., 525 and 620.

When the entire surface is required, add to the sum of the areas of the two bases, the area of the convex surface.

EXAMPLES.

1. What is the convex surface of a cylinder whose altitude is 24 feet and the diameter of whose base is 16 feet?

Ans., 1206.3744 sq. ft.

2. What is the entire surface of a right cylinder, the diameter of whose base is $2\frac{1}{2}$ feet, and the height 5 feet?

Ans., 49.0875 sq. feet.

3. The side of a cube is 19 feet; what is the area of its surface? *Ans.*, 2166 sq. feet.

4. The length of a parallelopiped is 8 feet, its breadth 3 feet and thickness 2 feet; to find the area of its entire surface. *Ans.*, 92 sq. feet.

5. Find the whole surface of a triangular prism, whose base is an equilateral triangle, having each of its sides equal to 18 inches, and altitude 20 feet.

Ans., 91.949 sq. feet.

§ 44. To find the convex or lateral surface of a right circular cone or regular pyramid.

[35] RULE: *The convex surface of a right circular cone or regular pyramid is equal to the perimeter of the base multiplied by one-half the slant height.*

When the entire surface is required, add the area of the base to the area of the convex surface.

EXAMPLES.

1. The circumference of the base of a right cone is 27.5 feet, and the slant height is 11 feet; required the convex surface. *Ans.*, 151.25 sq. feet.

2. Required the whole surface of a triangular pyramid, each side of its base being $5\frac{1}{2}$ feet, and its slant height $17\frac{1}{2}$ feet. *Ans.*, 157.4736 sq. feet.

3. What is the entire surface of a right pyramid, whose slant height is 27 feet, and the base a pentagon, of which each side is 25 feet? *Ans.*, 2762.798 sq. feet.

4. The circumference of the base of a right cone is 18.8496 feet, and the slant height 21 feet; required the convex surface. *Ans.*, 197.9208 sq. feet.

5. The slant height of a right cone is 20 feet and the diameter of the base 8 feet; required the convex surface. *Ans.*, 251.328 sq. feet.

§ 45. To find the convex or lateral surface of a frustum of a right circular cone or regular pyramid.

[36] RULE: *The convex surface of a frustum of a right circular cone or regular pyramid is equal to one-half the product of the slant height and the sum of the perimeters of the two bases.*

For Proof: See Ww. Geom., 570 and 661.

When the entire surface is required, add the sum of the areas of the two bases to the convex surface.

EXAMPLES.

1. In a frustum of a cone the circumferences of the bases are 22.75 and 15.5, and the slant height 26; what is the convex surface? *Ans.*, 497.25 sq. feet.

2. What is the convex surface of a frustum of a right cone, the diameters of the ends being 8 and 4 feet, and the length of the slant height 20 feet? *Ans.*, 376.992 sq. feet.

3. Required the surface of a hexagonal pyramid, one side of the lower base being $8\frac{1}{3}$ feet, and one side of the upper base being $3\frac{2}{3}$ feet, the slant height being $20\frac{5}{8}$ feet? *Ans.*, 733.5 sq. feet.

4. What is the convex surface of the frustum of a heptagonal pyramid, whose slant height is 55 feet, each side of the lower base 8 feet, and each side of the upper base 4 feet? *Ans.*, 2310 sq. feet.

§ 46. To find the convex surface of a cylindrical ungula.

[37] RULE: *The convex surface of a cylinder below an oblique cutting plane, is equal to the product of the circumference of the base and the half sum of the greatest and least elements of the required convex surface.*

Proof: Let *ABDC*, Fig. 17, be a cylinder cut by the plane *HNGM* oblique to the base. The cutting plane is passed perpendicular to the plane *ABD*; then *BH* must evidently be the longest element in the convex surface below the cutting plane, and *AG* the shortest one. *GH* will be the intersection of the plane *ABD* with the cutting plane; *G* and *H* being points common to the two planes. Through *O*, the middle point of *GH*, pass a horizontal plane; this will intersect the cutting plane in the line *MN*, and the cylinder in the circle *ENFM* equal to the base.

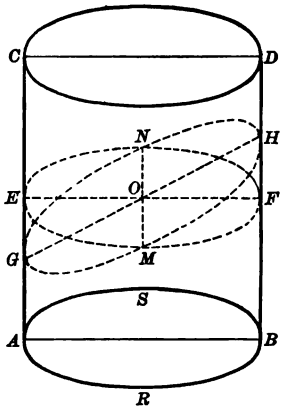

FIG. 17.

(Ww. Geom., 618.)

Now, the two cylindrical wedges *NMFH* and *NMEG* have the same edge *NM*, equal altitudes *EG* and *HF*, and the corresponding faces equal; therefore they are equal each to each, and

con. sur. $NEMG$ = con. sur. $NFMH$.

also, con. sur. $AB - GMNF$ = con. sur. $AB - GMNF$;

$\therefore$ con. sur. $AB - GMNF$ + con. sur. $NEMG$
= con. sur. $AB - GMNF$ + con. sur. $NFMH$;

but con. sur. $AB - GMNF$ + con. sur. $NEMG$
= con. sur. cylinder $ARBS - EMFN$ = circum. $AB \times BF$
= circum. $AB \times \frac{1}{2}(BH + AG)$;

and, con. sur. $AB - GMNF$ + con. sur. $NFMH$
= convex surface ungula $AB - GMHN$.

$\therefore$ convex surface ungula = circum. $AB \times \frac{1}{2}(BH + AG)$.

Q.E.D.

EXAMPLES.

1. What is the convex surface of a cylindrical ungula, the diameter of whose base is 16 feet, and the greatest and least heights are 17 and 14 feet?

Here, circum. $AB = 16 \times 3.1416 = 50.2656$.

Then, $50.2656 \times \frac{14 + 17}{2} = 779.1168$ sq. ft.

Ans.

2. What is the convex surface of a cylindrical ungula, the circumference of whose base is 21 feet, and the greatest and least heights are 13 and 8 feet? *Ans.*, 220.5 sq. ft.

§ 47. To find the surface of a regular polyhedron.

[38] RULE I: *The surface of a regular polyhedron is equal to the area of one of the faces multiplied by the number of the faces.*

As all the faces of a regular polyhedron are equal, the entire surface of the polyhedron must be equal to the area of one face, multiplied by the number of faces.

RULE II: *The surface of a regular polyhedron is equal to the square of one of the edges, multiplied by the surface of a similar polyhedron whose edge is unity.*

For, regular polyhedrons of the same number of faces are similar, and the entire surfaces of similar polyhedrons are proportional to the squares of homologous edges (Ww. Geom., 585). Therefore, the surface of a regular polyhedron, is to the surface of a similar polyhedron whose edge is unity, as the square of the edge of the given polyhedron is to 1.

Surfaces of regular Polyhedrons whose Edges are unity.

NAMES.	NO. OF FACES.	SURFACES.
Tetrahedron	4	1.7320508.
Hexahedron	6	6.0000000.
Octahedron	8	3.4641016.
Dodecahedron	12	20.6457288.
Icosahedron	20	8.6602540.

EXAMPLES.

1. What is the entire surface of a regular octahedron whose edges are each 8 feet? *Ans.*, 221.7025 sq. ft.

2. What is the entire surface of a regular tetrahedron whose edges are each 10 feet? *Ans.*, 173.20508 sq. ft.

3. What is the entire surface of a regular dodecahedron whose edges are each 12 feet? *Ans.*, 2972.985 sq. ft.

§ 48. To find the surface of a sphere.

[39] RULE: *The surface of a sphere is equal to the diameter multiplied by the circumference of a great circle; or is equal to the sum of the areas of four great circles.*

For Proof: See Ww. Geom., 763 and 761.

§ 49. To find the surface of a spherical zone.

[40] RULE: *The surface of a spherical zone is equal to the altitude of the zone multiplied by the circumference of a great circle of the sphere.*

For Proof: See Ww. Geom., 784.

EXAMPLES.

1. What is the convex surface of a sphere whose diameter is 4 feet? *Ans.*, 50.2656 sq. ft.

2. What is the convex surface of a sphere whose diameter is 6 feet? *Ans.*, 113.0976 sq. ft.

3. The diameter of a globe is 21 inches; what is the convex surface of that segment of it whose height is $4\frac{1}{2}$ inches? *Ans.*, 296.8812 sq. inches.

4. The diameter of a sphere being 42 inches, what is the area of the surface of a zone whose altitude is 9 inches? *Ans.*, 1187.5248 sq. inches.

5. If the diameter of the earth be 7912, what is the surface of the torrid zone, extending 23° 27′ 36″ on each side of the equator? *Ans.*, 78,293,218 sq. miles.

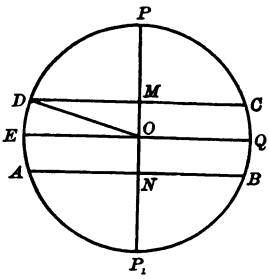

FIG. 18.

In Fig. 18 let *ABCD* represent the torrid zone; *EQ* representing the equator.

Then *OM*, or half the altitude of the zone, from the right-angled triangle *ODM*, is equal to $OD \times \sin. ODM$.

But *OD* is the radius, and the angle *ODM* is equal to the angle *EOD* or 23° 27′ 36″

$\therefore$ altitude $MN = 2\,OM = 2 \times$ radius $\times \sin$ (23° 27′ 36″).

§ 50. To find the surface of a lune.

[41] RULE: *The surface of a lune is equal to the surface of the sphere, multiplied by the ratio of the angle of the lune to four right angles.*

For Proof: See Ww. Geom., 766.

§ 51. To find the area of a spherical triangle.

[42] RULE: *The area of a spherical triangle is equal to the area of the trirectangular triangle, multiplied by the ratio of the spherical excess of the given triangle to one right angle.*

For Proof: See Ww. Geom., 771.

§ 52. To find the area of a spherical polygon.

[43] RULE: *The area of a spherical polygon is equal to the area of the trirectangular triangle, multiplied by the ratio of the spherical excess to one right angle.*

If three planes that are mutually perpendicular be passed through the centre of the sphere, they will divide the surface into eight trirectangular triangles; the area of the trirectangular triangle is, therefore, one-eighth the surface of the sphere. The spherical excess of a spherical triangle is the sum of its angles diminished by 180°; and the spherical excess of any spherical polygon is the sum of all its angles, diminished by two right angles taken as many times as the polygon has sides less two. It will be noticed in the last two rules, that the spherical excess itself is not the factor, but that the second factor in the product is the spherical excess divided by 90° or one right angle.

EXAMPLES.

1. Required the area of a triangle described on the surface of a sphere, whose diameter is 30 feet, the angles being 140°, 92°, and 68°.

Solution:

surface sphere $= 4\pi r^2 = 4 \times 3.1416 \times 15^2 = 2827.44$

surface trirect. triangle $= \frac{2827.44}{8} = 353.43$

spherical excess $= 300° - 180° = 120°$.

$\therefore$ area triangle $= \frac{120}{360} \times 353.43 = 471.24$ *Ans.*

2. What is the area of a lune on the surface of a sphere whose diameter is 32, the angle of the lune being 60°?

Ans., 536.1664.

3. What is the area of the surface of the earth between two meridians that are 36° apart; the diameter being 7912 miles? *Ans.*, 19,666,289.6 sq. miles.

4. What is the area of a triangle on a sphere whose diameter is 10 feet, if the angles are 55°, 60°, and 85°?

Ans., 8.7266 sq. ft.

5. What is the area of a spherical polygon of seven sides on a sphere whose diameter is 17 feet, the sum of the angles being 1080°? *Ans.*, 226.98 sq. ft.

6. The angles of a polygon on a sphere whose diameter is 10 feet, are 99° 22′ 47″; 119° 31′ 25′; 105° 37′ 13″; 140° 28′ 35″; 175° 0′ 0″; required the area of the polygon.

Ans., 43.633 sq. ft.

G. Mensuration of Volumes.

§ 53. To find the volume of a prism or cylinder.

[44] Rule: *The volume of any prism or cylinder is equal to the area of the base multiplied by the altitude.*

For Proof: See Ww. Geom., 542 and 623.

EXAMPLES.

1. What is the volume of a cylinder whose height is 30 feet and the circumference of its base 20 feet?

Ans., 954.96 cubic feet.

2. The diameter of the base of a cylinder is 16 inches, and the length 20 feet; what is the volume?

Ans., 27.9253 cu. ft.

3. The length of a parallelopipedon is 36 inches, the width 20 inches, and the depth 18 inches; how many cubic feet does it contain? *Ans.*, 7.5 cu. ft.

4. How many cubic feet are there in a cubical box, one of whose edges is 5 feet and 3 inches long?

Ans., $144\frac{5}{64}$ cu. ft.

5. What is the volume of a triangular prism whose length is 40 feet, when each side of the base is 18 inches long? *Ans.*, 38.9711 cu. ft.

§ 54. To find the volume of any pyramid or cone.

[45] RULE: *The volume of any pyramid or cone is equal to the area of its base multiplied by one-third of its altitude.*

For Proof: See Ww. Geom., 574 and 664.

EXAMPLES.

1. What is the volume of a cone, of whose base the diameter is 30 feet, and whose altitude is 36 feet?

Ans., 8482.32 cu. ft.

2. What is the volume of a regular hexagonal pyramid, each side of the base being 20 feet, and the altitude being 50 feet? *Ans.*, 17320.5066 cu. ft.

3. What is the volume of a square pyramid whose slant height is 22 feet, and each side of its base 10 feet?

Ans., 714.143 cu. ft.

4. If the circumference of the base of a cone is 60 feet, and its altitude 72 feet, what is the volume?

Ans., 6875.712 cu. ft.

5. The sides of the base of a triangular pyramid are 125, 173 and 216, and the altitude of the pyramid is 150, what is the volume? *Ans.*, 540,450 cu. ft.

6. What is the volume of a pentagonal pyramid, its altitude being 12 feet, and each side of its base 2 feet?

Ans., 27.5276 cu. ft.

§ 55. To find the volume of a frustum of a pyramid or cone.

[46] RULE: *The volume of a frustum of a pyramid or cone is equal to the sum of the areas of the two bases and a mean proportional between them, multiplied by one-third the altitude.*

For Proof: See Ww. Geom., 578 and 667.

EXAMPLES.

1. What is the volume of a frustum of a cone, the diameter of the lower base being 6 feet, and the diameter of the upper base being 4 feet, and the altitude of the frustum being 9 feet? *Ans.*, 179.0712 cu. ft.

2. What is the volume of the frustum of a regular hexagonal pyramid, whose altitude is 9 feet, each side of the lower base being 4 feet and each side of the upper base being 3 feet?

Solution: area upper base $= 2.598076 \times 3^2 = 23.382684$

area lower base $= 2.598076 \times 4^2 = 41.569216$;

hence, area mean proportional $= \sqrt{23.382684 \times 41.569216}$

$= 31.176912$;

$\therefore$ volume frustrum $= \frac{9}{3}(23.382684 + 41.569216 + 31.176912) = 288.386436$ cu. ft., *Ans.*

3. What is the volume of a frustum of a cone, the height being 5 feet, and the circumferences of the ends 8 feet and 12 feet? *Ans.*, 40.3205 cu. ft.

4. What is the volume of a frustum of a regular pentagonal pyramid, each side of the lower base being 18 inches, each side of the upper base being 12 inches, and the height 4 feet 6 inches? *Ans.*, 12.2583 cu. ft.

§ 56. To find the volume of a wedge.

A wedge is a volume bounded by a rectangle called the back, two trapezoids called faces, and two triangles called ends. The line in which the faces meet is called the edge. The two ends are taken equally inclined to the back. There are three cases.

Case 1. When the length of the edge is equal to the length of the back. In Fig. 19 *AB*, the edge, equals the length of the back *CDEF*. Here the ends are perpendicular to the back, and the wedge is a right prism, whose base is the triangle *BED*, and altitude $AB = CD$; hence the volume is equal to the area of *BED* multiplied by *AB*.

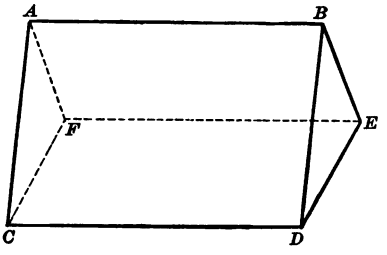

FIG. 19.

Case 2. When the length of the edge is less than the length of the back, Fig. 20. Through *M*, the middle point of *AB* the edge, pass a plane *MNR* perpendicular to the base, and intersecting it in the line *RN* parallel to *DE*. This plane will divide the wedge into two equal parts. Through *A* pass the plane *ASW* parallel to the plane *MNR*, and it will divide the left half of the wedge into a right triangular prism, $ASW - M$, and quadrangular pyramid $CSWF - A$. *AV* drawn perpendicular to *SW*,

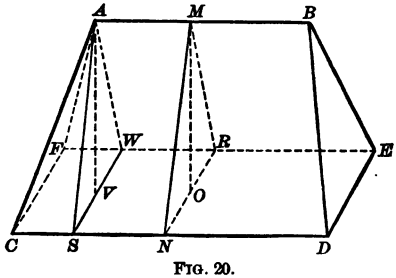

FIG. 20.

will also be perpendicular to the back of the wedge, and hence will be equal to the altitude of the wedge. Represent (AM) one-half the length of the edge by $\frac{1}{2} L$, the breadth (SW) by b, one-half the length of the back (CN) by $\frac{1}{2} l$, the altitude (AV) by a, and the half volume by $\frac{1}{2} V$; then

$$SN = AM = \tfrac{1}{2} L, \quad CS = \tfrac{1}{2} (l - L),$$

and $$\text{area } ASW = \tfrac{1}{2} (SW \times AV) = \tfrac{1}{2} ab;$$

then $$\text{vol. prism} = ASW \times AM = \tfrac{1}{4} abL;$$

$$\text{vol. pyramid} = CSWF \times \tfrac{1}{3} AV = \tfrac{1}{2} (l - L)\, b \times \tfrac{1}{3} a = \tfrac{1}{6} ab(l - L),$$

$$\therefore \text{ vol. half wedge} = \tfrac{1}{4} abL + \tfrac{1}{6} ab\,(l - L) = \tfrac{1}{4} abL + \tfrac{1}{6} abl - \tfrac{1}{6} abL = \tfrac{1}{12} abL + \tfrac{1}{6} abl = \tfrac{1}{12} ab\,(L + 2l),$$

and vol. entire wedge $= \frac{1}{6} ab\,(L + 2l) \cdots$ (1)

Case 3. When the length of the edge is greater than the length of the back. In this case the volume of the wedge will be equal to the difference between the volumes of the prism and pyramid,

and $$\text{vol. pyramid} = \tfrac{1}{6} ab\,(L - l);$$

hence, $$\text{entire volume wedge} = 2\,(\tfrac{1}{4} abL - \tfrac{1}{6} ab\,(L - l)) = \tfrac{1}{6} ab\,(L + 2l).$$

Therefore in all three cases the formula for the volume of the wedge will be the same, hence

[47] RULE: *Add twice the length of the back to the length of the edge; multiply the sum by one-sixth of the product of the height of the wedge and the breadth of the back; the final product will be the volume required.*

§ 57. To find the volume of a rectangular prismoid.

A rectangular prismoid is a frustum of a wedge. The two bases are rectangles, and the other four faces are trapezoids.

[48] RULE: *To find the volume of a prismoid, add together the areas of the two bases and four times the middle section, and multiply this sum by one-sixth the altitude.*

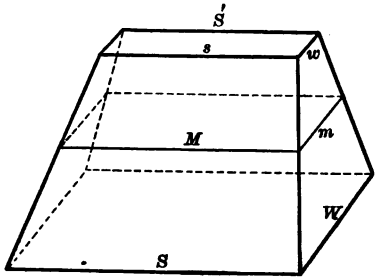

FIG. 21.

In Fig. 21, let S and W denote the length and width of the lower base; M and m the length and width of a section equidistant from the bases; s and w the corresponding dimensions of the upper base, and a the altitude of the prismoid.

If a plane be passed through the edges S and S', the prismoid will be divided into two wedges, whose bases are the bases of the prismoids and whose edges are S' and S.

The volume of the prismoid will be equal to the sum of these volumes of the two wedges; hence,

$$\text{vol. prismoid} = \tfrac{1}{6}\, aW\,(s + 2\,S) + \tfrac{1}{6}\, aw\,(S + 2\,s) \qquad (1)$$

or,
$$= \tfrac{1}{6}\, a\,(Ws + wS + 2\,WS + 2\,ws)$$
$$= \tfrac{1}{6}\, a\,[(Ws + wS + WS + ws) + WS + ws] \qquad (2)$$

But, since M is equally distant from S and s, we have,

$$2\,M = S + s, \text{ and } 2\,m = W + w;$$

hence, $4\,Mm = (S + s)\,(W + w) = SW + ws + Ws + wS.$

Substituting $4\,Mm$ for its value in equation (2), gives,

$$\text{vol. prismoid} = \tfrac{1}{6}\, a\,(WS + ws + 4\,Mm). \qquad \text{Q.E.D.}$$

EXAMPLES.

1. The length and breadth of the base of a wedge are 35 and 15 inches, the length of the edge 55 inches, and the altitude 18 inches; what is the volume?

Ans., 3.2552 cu. ft.

2. What is the volume of a wedge whose back is 18 feet by 9 feet, edge 20 feet, and altitude 6 feet?

Ans., 504 cu. ft.

3. What is the volume of a rectangular prismoid, the length and breadth of one end being 12 and 8 inches, and the corresponding sides of the other end 8 and 6 inches, the altitude being 5 feet? *Ans.*, 2.4537 cu. ft.

4. What is the volume of a log of hewn timber, whose ends are 30 by 27 inches, and 24 by 18 inches, and whose length is 48 feet? *Ans.*, 204 cu. ft.

5. One of the bases of a rectangular prismoid is 25 by 20, the other 15 by 10, and the altitude 12 feet; required the volume. *Ans.*, 3700 cu. ft.

§ 58. To find the volume of a prismatoid.

A prismatoid is a polyhedron whose bases are two polygons in parallel planes, the lateral faces being formed by joining the vertices of these bases so that each of the lateral edges in order forms a triangle with the preceding edge and one side of either base.

In Fig. 22, let $ABCDEFG$ be the prismatoid, the area of whose upper base ABC will be denoted by A_1, and the area of the lower base $DEFG$ by A_2. Pass a plane HLM parallel to the bases and equidistant from them. HO the intersection of this middle plane with ADC will be parallel to AC; HL, the intersection of this plane with DCE will be parallel to DE, &c. The lateral edges AD, DC, CE, &c., will be bisected at O, H, L, &c., and as the

altitude of the frustum is the perpendicular distance between the bases, it will also be bisected by the middle section.

Take any point P in the middle cross-section and draw lines from it to the corners of the prismatoid. The prismatoid is thus divided into two pyramids $PABC$ and $PDEFG$, and a series of polyhedrons of which $CPDE$ can be taken as a specimen.

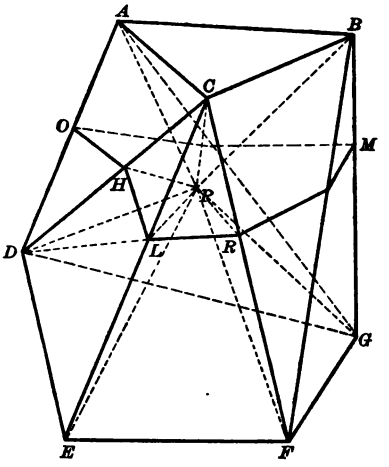

FIG. 22.

Let h be the altitude of the prismatoid, then $\frac{1}{2}h$ is the altitude of each of the pyramids $PABC$ and $PDEFG$,

hence, vol. $PABC = \frac{1}{6} h A_1$

and, vol. $PDEFG = \frac{1}{6} h A_3$.

Now draw PH, PL, and LD. The triangles CDL and DLE have equal bases and altitudes, and are therefore equal; and the pyramids $CPDE$ and $CPDL$ have their bases in the same plane and their vertices at the same point;

then, vol. $CPDE = 2$ vol. $CPDL$.

Now the pyramids, $CPDL$ and $CPHL$ have the same altitudes, and the base of the first double the base of the second. therefore,

vol. $CPDL = 2$ vol. $CPHL$,

hence, vol. $CPDE = 4$ vol. $CPHL$.

but, vol. $CPHL = \frac{1}{6} h \times$ area HLP,

$\therefore$ vol. $CPDE = \frac{4}{6} h \times$ area HLP.

In the same way the volume of the polyhedron $CPEF$ could be proved equal to $\frac{4}{6} h \times$ area PLR. And in like manner the volume of every such polyhedron is $\frac{4}{6} h$ multiplied by the area of its own portion of the cross-section. Hence if A_2 denote the entire area of the middle section, vol. prismatoid equals the sum of the pyramids whose bases are the bases of the prismatoid plus the pyramids whose bases are the lateral faces,

$$\therefore \text{volume prismatoid} = \tfrac{1}{6} hA_1 + \tfrac{1}{6} hA_3 + \tfrac{4}{6} hA_2$$
$$= \tfrac{1}{6} h (A_1 + 4A_2 + A_3). \qquad (1)$$

[49] RULE: *The volume of a prismatoid is equal to the sum of the areas of the two bases plus four times the middle section, and this last sum multiplied by one-sixth the altitude.*

§ 59. Equation (1) of the preceding article is a very general formula, and is frequently applied. It may be shown that the same formula will serve to determine the volume of every one of the forms hitherto given.

The *prismoid* is a prismatoid whose bases are polygons of the same number of sides having their corresponding sides parallel; the formula giving the volume of the prismoid is then the same as that for any prismatoid, for this reason the general formula is called the *prismoidal formula.*

The *wedge* is a prismatoid in which the lower base is a rectangle, and the upper base is a straight line parallel to the base. Then the area of the upper base is zero, and if this be substituted for A_1 in formula (1) § 58, the expression will reduce to equation (1) § 56.

The prismoidal formula may readily be applied to the case of prisms, cylinders, cones, pyramids, frustums,

spheres, &c. Thus, in a pyramid, we may regard the base as one extreme section, and the vertex whose area is zero as the other base. The area of a section midway between them is equal to one-fourth of the base; hence, four times the middle section is equal to the base. Thus according to the prismoidal rule, multiplying the sum of the base and the middle section, by one-sixth of the altitude, the same result is obtained as is given in Rule 45, § 54.

The prismoidal formula is of frequent application in determining the volumes of earth-work in railroad excavations and embankments. This formula gives the volumes between equidistant parallel vertical planes perpendicular to the axis of the road.

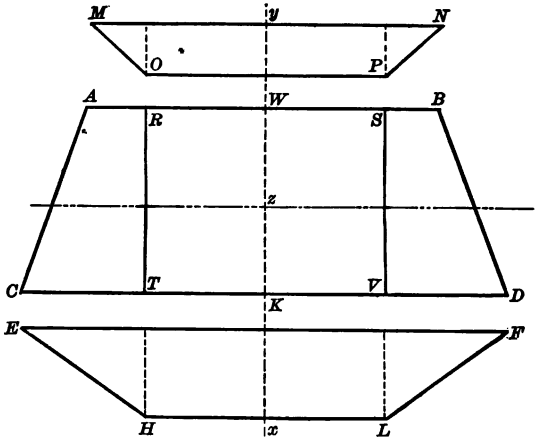

FIG. 23.

In Fig. 23, let xy represent the axis or centre line of the road, $RTVS$ the lower surface of the excavation which is horizontal, and $ABDC$ the horizontal projection of the upper surface: the volume being terminated by planes through W and K perpendicular to the centre line. Also,

let *MOPN* and *EHLF* represent the end vertical sections. Then by the prismoidal formula the volume will be equal to the area *EHLF* plus the area *MOPN* plus four times the area of a middle section at *z*, multiplied by one-sixth of the length *WK*.

§ 60. LEMMA. To find the dihedral angle between two adjacent faces of a regular polyhedron.

Conceive a sphere whose radius is unity as having its centre at the vertex of a polyhedral angle; then the surface of this sphere will intersect the faces of the polyhedral angle in arcs of great circles which together will form a regular spherical polygon. Then the sides of this polygon can be taken equal to the plane angles of the polyhedral angle, and its angles equal to the dihedral angles between the faces. Then the required dihedral angle is obtained by finding an angle of a regular spherical polygon whose sides are given.

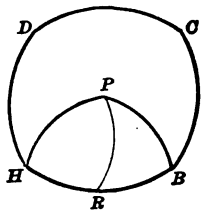

FIG. 24.

In Fig. 24, let *HBCD* represent the spherical polygon, and let *P* be the pole of a small circle passing through the points *H*, *B*, *C*, and *D*. Through *P* and the vertices *H*, *B*, *C*, and D pass arcs of great circles; the polygon will then be divided into as many equal isosceles triangles as the polygon has sides; and each angle thus formed at *P*, such as *HPB*, will be equal to 360°, divided by the number of sides in the polygon. Now draw an arc of a great

circle PR, perpendicular to HB: this will divide the base HB into two equal parts. If the number of the sides of the polygon is denoted by n, the angle HPR will be equal to $\frac{360°}{2n}$, or $\frac{180°}{n}$.

In the right-angled spherical triangle HPR, the base HR, and the angle HPR are known; hence,

$\sin PHR = \sec HR . \cos HPR$; (Ww. Trig., § 46, For. 41)

denoting the side HB by s, and the angle PHR by $\frac{1}{2}A$,

$$\sin \tfrac{1}{2}A = \sec \tfrac{1}{2}s . \cos \frac{180°}{n}.$$

In the case of the Tetrahedron,

$$\frac{180°}{n} = 60°, \text{ and } \tfrac{1}{2}s = 30°, \therefore A = 70° 31' 42.''$$

In the case of the Hexahedron,

$$\frac{180°}{n} = 60°, \text{ and } \tfrac{1}{2}s = 45°, \therefore A = 90°.$$

In the case of the Octahedron,

$$\frac{180°}{n} = 45°, \text{ and } \tfrac{1}{2}s = 30°, \therefore A = 109° 28' 18''.$$

In the case of the Dodecahedron,

$$\frac{180°}{n} = 60°, \text{ and } \tfrac{1}{2}s = 54°, \therefore A = 116° 33' 54''.$$

In the case of the Icosahedron,

$$\frac{180°}{n} = 36°, \text{ and } \tfrac{1}{2}s = 30°, \therefore A = 138° 11' 23''.$$

§ 61. To find the volume of a regular polyhedron.

[50] RULE: *The volume of a regular polyhedron is equal to the surface multiplied by one-third of the perpendicular let fall from the centre to one of the faces.*

The faces of a regular polyhedron are equally distant from a point within called the centre. Then if planes be passed through the centre and each of the edges of the polyhedron, they will divide the volume into as many equal right pyramids as the polyhedron has faces. The common vertex of the pyramids will be at the centre of the polyhedron, and each face of the polyhedron will be the base of a pyramid. The volume of each pyramid is equal to its base multiplied by one-third of its altitude; therefore the volume of the polyhedron will be equal to the sum of its faces multiplied by one-third of the perpendicular from the centre to a face.

It remains to find a formula for the perpendicular distance from the centre to a face of the polyhedron. The foot of this perpendicular will be the centre of the face.

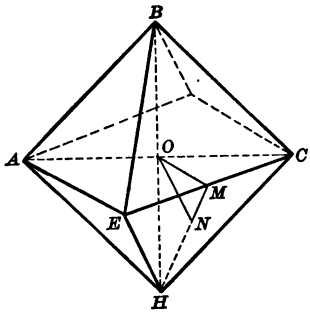

FIG. 25.

In Fig. 25, let *ABCHE* represent a regular polyhedron whose centre is at *O*, and let *ON* be the perpendicular from the centre to the face *HEC*. Then from *N*, the foot of the perpendicular, draw a line perpendicular to *EC*, one side of the face *HEC*, and connect the point *M* with the centre.

The triangle *ONM* is right-angled at *N*; the base *NM* is the apothem of the face *HEC*, the angle *OMN* is one-half the dihedral angle whose edge is *EC*, and the altitude *ON* is the altitude of the pyramid.

Then $ON = NM.\ \tan\ OMN.$ (Ww. Trig., § 14.)

Denoting the perpendicular P, the base by b, and the dihedral angle by A,

$$P = b.\ \tan\ \tfrac{1}{2}\ A;$$

but b is the apothem of a face; if, therefore, we represent the number of sides in that face by n, and the length of each side by s, from § 32,

$$b = \tfrac{1}{2}\, s.\cot\frac{180^\circ}{n};$$

$$\therefore\ P = \tfrac{1}{2}\, s.\cot\frac{180^\circ}{n}\tan\tfrac{1}{2}\, A.$$

The value of A is obtained by the preceding article.

According to the rule just given the volumes of all the regular polyhedrons whose edges are unity have been computed.

Volumes of regular polyhedrons when edges equal one.

NAMES.	FACES.	VOLUMES.
Tetrahedron	4	0.1178513
Hexahedron	6	1.0000000
Octahedron	8	0.4714045
Dodecahedron	12	7.6631189
Icosahedron	20	2.1816950

Similar pyramids are to each other as the cubes of their homologous edges; and as two regular polyhedrons of the same name can be divided into the same number of similar pyramids, they are to each other as the cubes of their edges. Therefore follows,

RULE II: *The volume of any regular polyhedron is equal to the cube of one of its edges multiplied by the volume of a similar polyhedron whose edge is unity.*

EXAMPLES.

1. One edge of a tetrahedron is 9 feet; what is its volume? *Ans.*, 85.9132 cu. ft.

2. What is the volume of a regular octahedron, one edge being 7 feet? *Ans.*, 161.691 cu. ft.

3. What is the volume of the regular dodecahedron, one edge being 6 inches? *Ans.*, 1655.2296 cu. in.

4. One edge of a regular icosahedron is 3 feet; what is its volume? *Ans.*, 58.9056 cu. ft.

5. The volume of an icosahedron is 58.905675 cu. ft.; what is the length of an edge? *Ans.*, 3 ft.

§ 62. To find the volume of a sphere.

[51] RULE: *The volume of a sphere is equal to the surface multiplied by one-third the radius; or the cube of the diameter multiplied by* $\frac{1}{6}\pi$.

For Proof: See Ww. Geom., 788 and 789.

EXAMPLES.

1. What is the volume of a sphere whose diameter is 17 inches? *Ans.*, 1.4886 cu. ft.

2. What is the volume of a sphere whose diameter is 3 feet 4 inches? *Ans.*, 19.3925 cu. ft.

3. What is the volume of the earth if it be a sphere 7912 miles in diameter? *Ans.*, 259,332,805,350 cu. miles.

§ 63. To find the volume of a spherical segment of one base.

[52] RULE: *The volume of a spherical segment of one base is equal to the sum of the product of the area of the base by one-half the height of the segment, and the product of the cube of the height by* $\frac{1}{6}\pi$.

For Proof: See Ww. Geom., 793.

§ 64. To find the area of a spherical segment of two bases.

[53] RULE: *The volume of a spherical segment of two bases is equal to the difference between two spherical segments each having a single base.*

For Proof: See Ww. Geom., 794.

EXAMPLES.

1. The radius of the base of a spherical segment of one base is 7 inches, and the height 4 inches; what is the volume? *Ans.*, 341.3872 cu. in.

2. The diameter of a sphere is 18 inches; what is the volume of a segment of one base cut from it whose height is 3 inches? *Ans.*, 226.1952 cu. in.

3. What is the volume of a segment of the earth cutting off a frigid zone, supposing the earth to be 7912 miles in diameter, and the polar circles to extend 23° 27′ 36″ from the poles? *Ans.*, 1,292,390,176 cu. miles.

4. What is the volume of a segment of a sphere whose lower base is 2 feet in diameter, upper base 1 foot 8 inches in diameter, and the distance between bases 4 inches? *Ans.*, 1566.6112 cu. in.

§ 65. To find the volume of a spherical sector.

[54] RULE: *The volume of a spherical sector is equal to the product of the area of the zone which forms its base by one-third the radius of the sphere.*

For Proof: See Ww. Geom., 791.

§ 66. To find the volume of a spherical pyramid.

[55] RULE: *The volume of a spherical pyramid is equal to the volume of the sphere, multiplied by the ratio of the base of the pyramid to the surface of the sphere.*

For Proof: See Ww. Geom., 775.

EXAMPLES.

1. What is the volume of a spherical pyramid when the radius of the sphere is 10.8, and the angles of its base are 78° 15′, 144° 30′ and 108° 15′? *Ans.*, 1106.61.

2. What is the volume of a spherical sector, the altitude of the zone forming its base being 2 feet, and the diameter of the sphere being $12\frac{1}{2}$ feet? *Ans.*, 327.25 cu. ft.

§ 67. Problems in Mensuration.

1. Two ships sail from the same port; at the end of three hours one has sailed due east 60 miles and the other due west 80 miles; how far are they then apart?

Ans., 100 miles.

2. In a right-angled triangle the sum of the hypothenuse and altitude is 240 yards, and the base is 80 yards; required the perpendicular. *Ans.*, $106\frac{2}{3}$ yards.

3. What is the diagonal of a square whose side is unity?

Ans., 1.4142 +.

4. What is the diagonal of a cube whose edge is unity?

Ans., 1.73205 +.

5. Two sides of a triangle are 3 yards and 8 yards long, and the included angle is 60°. What is the third side?

Ans., 7 yards.

6. Given the two sides of a triangle, 464.7 yards and 289.3 yards, and the included angle 87° 03′ 48″, what is the third side? *Ans.*, 534.66 yards.

7. If the diameter of the earth's orbit is 189,761,000, what is its circumference? *Ans.*, 596,151,764 miles.

8. What is the length of an arc of 25° in a circle whose radius is 44 rods? *Ans.*, 19.198 rods.

9. What is the length of an arc 11° 15′ in a circle whose diameter is 1234 feet? *Ans.*, 121.147 ft.

10. The chord of a whole arc is $50\frac{4}{5}$ yards, and the chord of half the arc is $33\frac{1}{3}$ yards; required the length of the arc. *Ans.*, 71.6 yards.

11. The chord of the whole arc is $48\frac{1}{2}$ feet, and the height of the arc $18\frac{1}{4}$ feet; what is the length of the arc?

Ans., 64.7667 ft.

12. What are the sides of a triangle containing 33.6 acres, in the proportion of 13, 14, and 15 chains?

Ans., 4.1082, 5.4776 and 8.2164 chains.

13. The base of an isosceles triangle is 24 yards, and the perpendicular 16 yards; what is the side of the inscribed square? *Ans.*, 9.6 yards.

14. The base of a triangle is 324 yards, and the other two sides are 264 and 162 yards; what is the length of the perpendicular to the base from the vertex of the opposite angle? *Ans.*, 131.2613 yards.

15. The altitude of a right-angled triangle divides the hypothenuse into two parts, 7.2 and 16.2 feet long; what is the area? *Ans.*, 126.36 sq. yards.

16. The altitude of an equilateral triangle is 8.5; what is the area? *Ans.*, 41.71 acres.

17. The three sides of a triangle are 10.2, 13.6, and 17 feet; what is the area? *Ans.*, 69.36 sq. ft.

18. The base of a triangle is 20 feet, and its altitude 18 feet. It is required to draw a line parallel to the base, so as to cut off a trapezoid containing 80 square feet. What is the length of the line of section? *Ans.*, 14.907 ft.

19. The sides of a quadrilateral field are 690 yards, 467 yards, 359 yards, and 428 yards; also, the angle contained between the first and second sides is 57° 30′, and the angle between the third and fourth sides 96° 42′; what is the area of the field? *Ans.*, 212184 square yards.

20. The area of a parallelogram is 120 square feet, and two sides are 12 and 14 feet; what are the diagonals? *Ans.*, 24 and 10.2 yds.

21. In making an excavation for a railroad, the cut at the base must be 24 feet wide; if the excavation is 40 feet deep, what must be the width of the cutting at the top, if the sides slope at 65°? *Ans.*, 61.305 ft.

22. There are two regular pentagons, one inscribed in a circle and the other circumscribed about it; and the difference of the areas of the pentagons is 100 square inches. What is the radius of the circle? *Ans.*, 8.926 inches.

23. Required the length of the side of a regular hexagon containing one acre? *Ans.*, 7.8475 rods.

24. How many pieces, each 4 inches square may be cut from a regular decagon whose side is 12 inches?

Ans., 69.2478 pieces.

25. Of two concentric circles the area of the outer one is 100 acres, and the diameter of the inner one is two-thirds of the diameter of the outer one; what is the area of the circular ring? *Ans.*, 555.55 sq. chains.

26. The circumference of a circular walk is 64 rods; what is the side of a square containing the same area?

Ans., 18.0544 rods.

27. What is the side of a square inscribed in a semicircle whose radius is r? *Ans.*, $\frac{2}{5} r \sqrt{5}$.

28. The length of a sector of a given circle is 16 yards; and the angle of the sector at the centre is $\frac{1}{6}$ of a right angle. What is the area of the sector? *Ans.*, 488.9 sq. yds.

29. What is the area of a segment of a circle, whose chord is 16 chains, and diameter of circle is $16\frac{2}{3}$ chains?

Ans., 70.2222 sq. chains.

30. What is the area of a segment of a circle whose height is 3 feet, and the diameter of the circle 8 feet?

Ans., 17.2198 sq. ft.

31. The parallel sides of a trapezoid are 20 and 12 feet, and the other sides are 15 and 17 feet; what is the area?

Ans., 240 sq. ft.

32. What is the convex surface of a right cone whose altitude is 12 feet and slant height 12 feet?

Ans., 73.5 sq. ft.

33. The vertical ends of a hollow trough are parallel equilateral triangles, each side being one yard, the ends having one side horizontal. If the length of the trough is 6 yards, what is its capacity? *Ans.*, 2.598 cu. yds.

34. If a cylindrical tank whose length is 30 feet and diameter 10 feet is placed with its axis horizontal; how many gallons of oil does it contain when the depth of the oil is 7 feet 9 inches?

35. How high above the surface of the earth must a person be raised in order to see one-third of its surface?

Ans., height = diameter.

36. If a cone whose height is 20 inches is divided into three equal parts by two planes parallel to the base; what will be the altitudes of the three parts?

Ans., 13.8672, 3.6044, and 2.5284 in.

37. The diameter of the base of a spherical segment is 20 inches, and the height 6 inches; how many gallons of water will it hold, each gallon containing 282 cubic inches?

Ans., 3.743 gallons.

38. Required the volume of a spherical segment, the diameters of both bases being 3 feet, and the altitude 4 feet? *Ans.*, 61.7848 cu. ft.

39. The altitude of a cone is 10 inches, and the radius of the base 5 inches. At what distance from the base must a plane pass parallel to the base, so as to cut off a frustum whose capacity is 20 cubic inches? *Ans.*, 0.2614 in.

40. Find the radius of a sphere which is equal to the sum of two spheres whose radii are 3 yards and 6 yards?

Ans., $3\sqrt[3]{9}$ yds.

41. A plane parallel to the base of a hemisphere bisects the radius perpendicular to the base; what is the ratio of the parts of the hemisphere? *Ans.*, 5 : 11.

42. Through a cylinder whose altitude is 15, and diameter of base is 20, a plane is passed parallel to the axis bisecting the basal radius perpendicular to it; what is the area of the section? *Ans.*, $150\sqrt{3}$.

43. Find the volume of a prismoid whose altitude is 15 yards, base 36 square yards, and a lower basal edge is to its parallel top edge as 3 to 2? *Ans.*, 380 cu. yds.

44. What is the area of a regular spherical polygon of eight sides on a sphere whose diameter is 30 yards, each angle of the polygon being 140°? *Ans.*, 157.08 sq. yds.

45. If a cask which is composed of two equal conic frustums joined together at their larger bases, have its bung diameter 28 inches, the head diameter 20 inches, and the length 40 inches; how many gallons of wine will it contain? *Ans.*, 79.0613 gallons.

46. What is the volume of a stick of hewn timber, whose ends are 30 inches by 27, and 24 inches by 18, and the length 24 feet? *Ans.*, 102 cu. ft.

47. What is the volume of a regular icosahedron, whose edge is 20? *Ans.*, 17453.56.

48. What is the volume of a spherical pyramid, when the radius of the sphere is 8.8 feet, and the angles of the spherical polygon are 106° 30′, 120° 10′, 150° 15′, and 112° 5′? *Ans.*, 511.433 cu. ft.

49. The capacity of a cylinder is a cubic feet, and its convex surface is b square feet. Required the dimensions of the cylinder? *Ans.*, $\frac{2a}{b}$ = radius, $\frac{b^2}{4a\pi}$ = altitude.

50. If a heavy sphere, whose diameter is 4 inches, be let fall into a conical glass full of water, whose diameter is 5 and altitude 6 inches, it is required to determine how much water will run over. *Ans.*, 26.272 cu. in.

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